

Approaching Capacity at High-Rates with Iterative Hard-Decision Decoding

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Abstract—A variety of low-density parity-check (LDPC) ensembles have now been observed to approach capacity with message-passing decoding. However, all of them use soft (i.e., non-binary) messages and a posteriori probability (APP) decoding of their component codes. In this paper, we analyze a class of spatially-coupled generalized LDPC codes and observe that, in the high-rate regime, they can approach capacity under iterative hard-decision decoding. These codes can be seen as generalized product codes and are closely related to braided block codes.

Index Terms—GLDPC codes, density evolution, product codes, braided codes, syndrome decoding

I. INTRODUCTION

In his groundbreaking 1948 paper, Shannon defined the capacity of a noisy channel as the largest information rate for which reliable communication is possible [1]. Since then, researchers have spent countless hours looking for ways to achieve this rate in practical systems. In the 1990s, the problem was essentially solved by the introduction of iterative soft decoding for turbo and low-density parity-check (LDPC) codes [2], [3], [4]. Although their decoding complexity is significant, these new codes were adopted quickly in wireless communication systems where the data rates were not too large [5], [6]. In contrast, complexity issues have slowed their adoption in very high-speed systems, such as those used in optical and wireline communication.

In this paper, we consider an ensemble of spatially-coupled generalized LDPC (GLDPC) codes based on t -error correcting block codes. For the binary symmetric channel (BSC), we show that the redundancy-threshold tradeoff of this ensemble, under iterative *hard-decision decoding*, scales optimally in the high-rate regime. To the best of our knowledge, this is the first example of an iterative hard-decision decoding (HDD) system that can approach capacity. Extrinsic information (EXIT) curves are also used to observe the phenomenon of threshold saturation for this decoder. It is interesting to note that iterative HDD of product codes was first proposed well before the recent revolution in iterative decoding but the performance gains were limited [7]. Iterative decoding of product codes became competitive only after the advent of iterative soft decoding based on the Turbo principle [8].

Our choice of ensemble is motivated by variations product codes now used in optical communications [9]. These, in turn, are also closely related to braided block codes [10], [11]. Our focus on HDD of generalized product codes based on t -error correcting block codes is similar to recent work by Justesen,

who has been exploring a variety of coding system for optical communication systems [12], [13]. The main difference is that the spatially-coupled GLDPC ensemble can be analyzed via DE, which allows one to rigorously account for spatial coupling and miscorrection. From DE, one can observe that iterative hard-decision decoding (HDD) can approach capacity.

II. BASIC ENSEMBLE

Let \mathcal{C} be an (n, k, d) binary linear code that can correct all error patterns of weight at most t (i.e., $d \geq 2t + 1$). For example, one might choose \mathcal{C} to be an primitive BCH code with parameters $(2^\nu - 1, 2^\nu - \nu t - 2, 2t + 2)$. Now, we consider a GLDPC ensemble where every bit node satisfies two code constraints defined by \mathcal{C} .

Definition 1. The (\mathcal{C}, m) GLDPC ensemble is defined by a Tanner graph, denoted by $\mathcal{G} = (\mathcal{I} \cup \mathcal{J}, \mathcal{E})$, with a set \mathcal{I} of $N = \frac{mn}{2}$ degree-2 bit nodes and a set \mathcal{J} of m degree- n code-constraint nodes defined by \mathcal{C} . A uniform random permutation is used to connect the mn edges from the bit nodes to the constraint nodes.

A. Decoder

It is well-known that GLDPC codes perform well under iterative soft decoding [8]. The main drawback is that a posteriori probability (APP) decoding of the component codes can be very complex. For this reason, we consider message-passing decoding based on bounded-distance decoding (BDD) of the component codes. Let $d_H(\cdot, \cdot)$ be the Hamming distance and $D_i : \{0, 1\}^n \rightarrow \{0, 1\}$ be the bit-level mapping implied by BDD, which is given by

$$D_i(v) = \begin{cases} c_i & \text{if } c \in \mathcal{C} \text{ satisfies } d_H(c, v) \leq t \\ v_i & \text{if } d_H(c, v) > t \text{ for all } c \in \mathcal{C}. \end{cases}$$

More generally, our analysis holds for symmetric decoders where the mapping $D_i(\cdot)$ satisfies $D_i(v \oplus c) = D_i(v) \oplus c_i$ for all $c \in \mathcal{C}$ and $i = 1, \dots, n$. For example, all syndrome decoders have this property.

Decoding proceeds by passing binary messages along the edges connecting the variable and constraint nodes. Let $r_i \in \{0, 1\}$ denote the received channel value for variable node i and $\mu_{i \rightarrow j}^{(\ell)}$ be the binary message from the i -th variable node to the j -th constraint node in the ℓ -th iteration. For simplicity, we assume no bit appears twice in a constraint and let $\sigma_j(k)$

be the index of the variable node connected to the k -th socket of the j -th constraint. Let j' be the other neighbor of the i -th variable node. Suppose that $\sigma_j(k) = i$. Then, the iterative decoder is defined by the recursion

$$\mu_{i \rightarrow j'}^{(\ell+1)} \triangleq D_i \left(v_{j \rightarrow i}^{(\ell)} \right), \quad (1)$$

where, for the i -th variable node and the j -th constraint node, the candidate decoding vector is $v_{j \rightarrow i}^{(\ell)} = (\mu_{\sigma_j(1) \rightarrow j}^{(\ell)}, \mu_{\sigma_j(2) \rightarrow j}^{(\ell)}, \dots, \mu_{\sigma_j(n) \rightarrow j}^{(\ell)})$ except that the k -th entry is replaced by r_i .

B. Density Evolution

The iterative decoding performance of GLDPC codes can be analyzed via density evolution (DE) because, for a randomly chosen bit node, any fixed-depth neighborhood in the Tanner graph is a tree with high probability as $m \rightarrow \infty$. For HDD of the component codes, this DE can be written as a one-dimensional recursion.

If we assume that the component decoders are symmetric, then it suffices to consider the case where the all-zero codeword is transmitted over a BSC with error probability p . Let $x^{(\ell)}$ be the error probability of the hard-decision messages passed from the variable nodes to the constraint nodes after ℓ iterations. For an arbitrary symmetric decoder, let $P_n(i)$ be the probability that a randomly chosen bit is decoded incorrectly when it is initially incorrect and there are i random errors in the other $n-1$ positions. Likewise, let $Q_n(i)$ be the probability that a random chosen bit is decoded incorrectly when it is initially correct and there are i random errors in the other $n-1$ positions. Then, for the (\mathcal{C}, m) GLDPC ensemble, the DE recursion implied by (1) is defined by $x^{(\ell+1)} = f_n(x^{(\ell)}; p)$ with $\bar{p} = 1 - p$ and

$$f_n(x; p) \triangleq \sum_{i=0}^{n-1} \binom{n-1}{i} x^i (1-x)^{n-i-1} (pP_n(i) + \bar{p}Q_n(i)) \quad (2)$$

The quantities $P(i)$ and $Q(i)$ can be written in terms of the number of codewords, A_l , of weight l in \mathcal{C} . Since the decoding regions of each codeword are disjoint, one finds that, for $t \leq i \leq n-1$,

$$P_n(i) = 1 - \sum_{\delta=0}^t \sum_{l=i+1-\delta}^{i+1+\delta} \frac{\frac{n-l}{n} A_l \binom{l}{\frac{1}{2}(l+i+1-\delta)} \binom{n-l-1}{\frac{1}{2}(i-l-1+\delta)}}{\binom{n-1}{i}}, \quad (3)$$

and $P_n(i) = 0$ for $0 \leq i \leq t-1$. Similarly, one finds that

$$Q_n(i) = \sum_{\delta=0}^t \sum_{l=i-\delta}^{i+\delta} \frac{\frac{l}{n} A_l \binom{l-1}{\frac{1}{2}(l+i-\delta)} \binom{n-l}{\frac{1}{2}(i-l+\delta)}}{\binom{n-1}{i}} \quad (4)$$

for $t+1 \leq i \leq n-1$, and $Q_n(i) = 0$ for $0 \leq i \leq t$.

Similar to DE for LDPC codes on the BEC [14, pp. 95–96], there is a compact characterization of the hard-decision decoding threshold p^* . The idea is that the successful decoding condition $f_n(x; p) < x$ provides a natural

lower bound on the noise threshold that can be rewritten as $p[f_n(x; 1) - f_n(x; 0)] + f_n(x; 0) < x$ to give

$$p^* = \inf_{x \in (0, 1)} \frac{x - f_n(x; 0)}{f_n(x; 1) - f_n(x; 0)}.$$

It is also worth noting that essentially the same recursion can be used for a BEC with erasure probability p . In this case, $Q_n(i) = 0$ and one redefines $P_n(i)$ to be the probability that a randomly chosen bit is not recovered when it is initially erased and there are i random erasures in the other $n-1$ positions.

III. SPATIALLY-COUPLED ENSEMBLE

Now, we consider a spatially-coupled GLDPC ensemble where every bit node satisfies two code constraints defined by \mathcal{C} . Similar to the definition introduced in [15], the spatially-coupled GLDPC ensemble (\mathcal{C}, m, L, w) is defined as follows.

Definition 2. The (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensemble can be constructed from the underlying Tanner graph with n degree-2 bit nodes and 2 degree- n code-constraint nodes defined by \mathcal{C} . The Tanner graph of a spatially-coupled GLDPC contains L positions of bit nodes and $L+w-1$ positions of code-constraint nodes. At each position, there are $\frac{mn}{2}$ degree-2 bit nodes and m degree- n code-constraint nodes. The (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensemble is constructed so that (i) the 2 edges from a bit node at position i are uniformly and independently connected to code-constraint nodes at positions $[i, \dots, i+w-1]$ and (ii) the n edges of a code-constraint node at position i are independently connected to bit nodes at positions $[i-w+1, \dots, i]$.

A. Density Evolution

To derive the DE update equation of the (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensemble, we let $x_i^{(\ell)}$ be the average error probability of hard-decision messages emitted by bit nodes at position i after the ℓ -th iteration. By the second property of the (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensemble, the average error probability of all inputs to a code-constraint node at position i is $\bar{x}_i^{(\ell)} = \frac{1}{w} \sum_{j=0}^{w-1} x_{i-j}^{(\ell)}$. From the first property, it follows that $x_i^{(\ell)} = \frac{1}{w} \sum_{k=0}^{w-1} f_n(\bar{x}_{i+k}^{(\ell-1)}; p)$, where $f_n(x; p)$ is defined in (2) and $x_i^{(\ell)} = 0$ for $i \notin [1, L]$. Therefore, the DE update for this ensemble is given by

$$x_i^{(\ell+1)} = \frac{1}{w} \sum_{j=0}^{w-1} f_n \left(\frac{1}{w} \sum_{k=0}^{w-1} x_{i-j+k}^{(\ell)}; p \right). \quad (5)$$

IV. BCH COMPONENT CODES

In the remainder of this paper, the $(n, k, 2t+1)$ binary primitive BCH code as well as its even weight $(n, k-1, 2t+2)$ subcode will be used as the component codes for both (\mathcal{C}, m) GLDPC and (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensembles. When the exact weight spectrum is known, one can use it to compute $P_n(i)$ and $Q_n(i)$ by (3) and (4), respectively.

Otherwise, we use the following binomial approximation for binary primitive BCH codes with large n [16],

$$A_l = \begin{cases} 2^{-\nu t} \binom{n}{l}, & d \leq l \leq n-d \\ 1, & l = 0, l = n \\ 0, & \text{otherwise,} \end{cases} \quad (6)$$

where $n = 2^\nu - 1$. Given these binomial approximation, we are able to simplify $P_n(i)$ and $Q_n(i)$ based on similar calculations in [16, Appendix A]. The approximation of $P_n(i)$ and $Q_n(i)$ for a primitive BCH code allows the simplified expressions

$$\hat{P}_n(i) = 1 - \frac{(n+1)^{-t}}{i+1} \sum_{\delta=0}^t \sum_{\alpha=0}^{\delta} \alpha \binom{n-i-1}{\delta-\alpha} \binom{i+1}{\alpha}, \quad (7)$$

and

$$\hat{Q}_n(i) = \sum_{\delta=1}^t \frac{(n+1)^{-t}}{n-i} \left(\delta \binom{n}{\delta} - \sum_{\alpha=0}^{\delta} \alpha \binom{i}{\alpha} \binom{n-i}{\delta-\alpha} \right). \quad (8)$$

By substituting (7) and (8) into (2), one can get the DE recursion of the (\mathcal{C}, m) GLDPC ensemble. Likewise, substituting (7) and (8) into (5), we have the DE recursion of the (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensemble.

For the $(n, k-1, 2t+2)$ even-weight subcode of an $(n, k, 2t+1)$ primitive BCH code, we approximate the number of codewords, denoted by \tilde{A}_l , by (6) when l is even, and set $\tilde{A}_l = 0$ when l is odd. Let $\tilde{P}_n(i)$ and $\tilde{Q}_n(i)$ be the miscorrection probabilities corresponding to the even-weight BCH subcode. Then, one can get the DE recursions for the (\mathcal{C}, m) GLDPC ensemble and the (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensemble can be obtained by using (2) and (5), respectively.

A. High-Rate Scaling

In [13], Justesen has analyzed the asymptotic performance of long product codes under the assumption that the decoder has no miscorrection. Using the random graph argument, a recursion for the ‘‘Poisson parameter’’ is obtained. That recursion leads to a threshold, for successful decoding, on the average number of errors in each code constraint. In this section, we obtain a similar recursion as the scaled limit, as $n \rightarrow \infty$, of our DE analysis. The novel contribution is that this approach rigorously accounts for miscorrection. We first introduce a few lemmas that simplify the development.

Lemma 3. $P_n(i) = 0$ for $0 \leq i \leq t-1$ and the upper bound $P_n(i) \leq 1$ for $t \leq i \leq n-1$ is sharp as $n \rightarrow \infty$.

Lemma 4. $Q_n(i) = 0$ for $0 \leq i \leq t$. For any $\epsilon > 0$, there exists an $N > 0$ such that, for all $n \geq N$ and any $\beta \in (0, 1)$, it follows for $0 \leq i \leq \lfloor \beta(n-1) \rfloor$ that

$$\hat{Q}_n(i) \leq \frac{1}{n(1-\beta)(t-1)!} + \frac{\epsilon}{n(1-\beta)},$$

and

$$\tilde{Q}_n(i) \leq \begin{cases} \frac{1}{n(n+1)(1-\beta)} \left(\frac{1}{(t-2)!} + \epsilon \right), & \text{if } i+t \text{ is odd} \\ \frac{1}{n(1-\beta)} \left(\frac{1}{(t-1)!} + \epsilon \right), & \text{if } i+t \text{ is even.} \end{cases}$$

Consider the DE recursion for the (\mathcal{C}, m) GLDPC ensemble based on (2). Let $p = \frac{\lambda^{(0)}}{n-1}$ with fixed $\lambda^{(0)}$ scale with n , and let $\Lambda_n^{(\ell)} = (n-1)x^{(\ell)}$. The scaled recursion for $\Lambda_n^{(\ell)}$ is given by

$$\Lambda_n^{(\ell+1)} = (n-1)f_n \left(\frac{\Lambda_n^{(\ell)}}{n-1}; \frac{\lambda^{(0)}}{n-1} \right),$$

where $\Lambda_n^{(0)} = \lambda^{(0)}$. Since $P_n(i) = Q_n(i) = 0$ for $0 \leq i \leq t-1$ and Lemma 3 shows $P_n(i) \rightarrow 1$ for $t \leq i \leq n-1$, we find that

$$\Lambda_n^{(\ell+1)} \leq \sum_{i=t}^{n-1} \binom{n-1}{i} \left(\frac{\Lambda_n^{(\ell)}}{n-1} \right)^i \left(1 - \frac{\Lambda_n^{(\ell)}}{n-1} \right)^{n-1-i} \times \left(\lambda^{(0)} (1 - Q_n(i)) + (n-1)Q_n(i) \right). \quad (9)$$

Lemma 5. Denote the tail probability of the Poisson distribution with mean λ by $\phi(\lambda; t) \triangleq \sum_{i=t+1}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} = 1 - \sum_{i=0}^t \frac{\lambda^i}{i!} e^{-\lambda}$. For any $\lambda \geq 0$, define $\hat{n} \triangleq n-1$ and $\alpha_n \triangleq \lambda \hat{n}^{-1}$. Then,

$$\lim_{n \rightarrow \infty} \sum_{i=t+1}^{\hat{n}} \binom{\hat{n}}{i} \alpha_n^i (1 - \alpha_n)^{\hat{n}-i} \hat{Q}_n(i) = 0, \quad (10)$$

and

$$\lim_{n \rightarrow \infty} \sum_{i=t+1}^{\hat{n}} \binom{\hat{n}}{i} \alpha_n^i (1 - \alpha_n)^{\hat{n}-i} \hat{Q}_n(i) \leq \frac{\phi(\lambda; t)}{(t-1)!}, \quad (11)$$

where $\hat{Q}_n(i)$ is defined in (8).

Sketch of proof: For any finite n , the RHS of (10) and (11) can be separated into two parts, summation over $[t+1, \lfloor \beta n \rfloor]$ and summation over $[\lfloor \beta n \rfloor + 1, \infty]$, where $\beta \in (0, 1)$. By applying the results in Lemma 4, both (10) and (11) can be obtained. ■

Define $\lambda^{(\ell)} \triangleq \limsup_{n \rightarrow \infty} \Lambda_n^{(\ell)}$. Substituting the approximation $\hat{Q}_n(i)$ for $Q_n(i)$ in (9) and applying Lemma 5 allows us to obtain

$$\lambda^{(\ell+1)} \lesssim \lambda^{(0)} \phi(\lambda^{(\ell)}; t-1) + \frac{1}{(t-1)!} \phi(\lambda^{(\ell)}; t). \quad (12)$$

Remark 6. For any $n < \infty$, $\frac{n\lambda^{(0)}}{n-1}$ can be seen as the average number of initial error bits attached to a code-constraint node, and $\frac{n\Lambda_n^{(\ell)}}{n-1}$ can be considered as the average number of error bits per code-constraint node after ℓ -th iteration. From $\frac{n\lambda^{(0)}}{n-1} \rightarrow \lambda^{(0)}$ and $\limsup_{n \rightarrow \infty} \frac{n\Lambda_n^{(\ell)}}{n-1} \rightarrow \lambda^{(\ell)}$, it follows that the scaled recursion (12) tracks the upper bound of the average number of error bits per code-constraint node with iteration. Also, while the derivation of (12) involves an approximation before the limit, we believe that final expression is exact.

When the even-weight BCH subcode is used as the component code, we can have an upper bound of $\hat{Q}_n(i)$ similar to (11). Let

$$\psi(\lambda; t) \triangleq \frac{1 + e^{-2\lambda}}{2} - \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \frac{\lambda^{2i}}{(2i)!} e^{-\lambda}$$

be the tail probability for the even terms of a Poisson distribution with mean λ and started at $t + 1$, and

$$\varphi(\lambda; t) \triangleq \frac{1 - e^{-2\lambda}}{2} - \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \frac{\lambda^{(2i+1)}}{(2i+1)!} e^{-\lambda}$$

be the tail probability for the odd terms of a Poisson distribution with mean λ and started at $t + 1$. With the same definitions of λ , α_n and \hat{n} in Lemma 5, it can be shown that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=t+1}^{\hat{n}} \binom{\hat{n}}{i} \alpha_n^i (1 - \alpha_n)^{\hat{n}-i} \hat{n} \tilde{Q}_n(i) \\ \leq \begin{cases} \frac{1}{(t-1)!} \psi(\lambda; t) & \text{if } t \text{ is even,} \\ \frac{1}{(t-1)!} \varphi(\lambda; t) & \text{otherwise.} \end{cases} \end{aligned}$$

Then, the DE recursion of the GLDPC ensemble with the even-weight BCH subcode can be obtained by using (12) as well. When t is even, replace $\phi(\lambda; t)$ in (12) with $\psi(\lambda; t)$, or otherwise, replace $\phi(\lambda; t)$ with $\varphi(\lambda; t)$.

For the spatially-coupled GLDPC ensemble, let $\lambda^{(0)}$ be the average number of initial error bits per component code, and $\lambda_i^{(\ell)}$ be the average number of errors per component code at position i after ℓ -th iteration. Define $\bar{\lambda}_i^{(\ell)} \triangleq \frac{1}{w} \sum_{j=0}^{w-1} \lambda_{i+j}^{(\ell)}$ as the averaged number of error inputs to a code-constraint node at position i . Similar to (12), the recursion for spatially-coupled ensemble is

$$\lambda_i^{(\ell+1)} \lesssim \frac{1}{w} \sum_{j=0}^{w-1} \lambda^{(0)} \phi\left(\bar{\lambda}_{i+j}^{(\ell)}; t-1\right) + \frac{1}{(t-1)!} \phi\left(\bar{\lambda}_{i+j}^{(\ell)}; t\right). \quad (13)$$

Also, replacing $\phi(\lambda; t)$ in (13) with $\psi(\lambda; t)$ or $\varphi(\lambda; t)$ according to the parity of t gives us the DE recursion of the spatially-coupled ensemble with even-weight BCH subcodes. Finally, we observe that the second term in (13) is upper bounded by $\frac{1}{(t-1)!}$ times the first term. This implies that miscorrection has a negligible affect on the threshold as $t \rightarrow \infty$.

V. EXIT FUNCTION BOUNDS

For coding systems based on iterative decoding, one can ask if the gap to capacity is due to the code or to the decoder. Fortunately, this question can now be tackled using the theory of extrinsic information (EXIT) functions. For example, if we ignore the effect of miscorrection and consider the natural hard-decision peeling decoder for the (\mathcal{C}, m) ensemble based on BCH codes, then it is easy to see that at most mt errors can be corrected using BDD. To achieve this result, it must happen that each code corrects exactly t errors. If some codes decode with fewer than t errors, then there is an irreversible loss of error-correcting potential. Since there are $\frac{nm}{2}$ code bits per code constraint, normalizing this number shows that at most $2t$ errors per code constraint are correctable.

In this section, we apply the theory of EXIT functions for the BEC to understand the HDD performance of our spatial-coupled LDPC ensemble. The idea is that the DE recursion, when miscorrection is ignored, is equivalent to the DE recursion for the erasure channel with component codes that correct

t erasures. For this model, we consider the scaled DE recursion without miscorrection, $\lambda^{(\ell+1)} = \lambda^{(0)} \phi(\lambda^{(\ell)}; t-1)$.

One can use EXIT functions to rigorously compute the MAP threshold for such a system on the erasure channel. We conjecture that the spatially-coupled version of this ensemble will have a BEC iterative decoding threshold that equals this MAP threshold. Based on this, we can predict the threshold of spatially-coupled HDD decoding, in the absence of miscorrection, because the peeling decoders for the two problems (i.e., BEC and HDD on BSC) have the same threshold. Taking this approach, we use the scaled BEC-MAP EXIT function $h^{\text{BMAP}}(\lambda) = (\phi(\lambda; t-1))^2$ to compute (e.g., see [14]) the scaled trial entropy

$$\begin{aligned} P(\lambda) &= \int_0^\lambda h^{\text{BMAP}}(z) \frac{d}{dz} \frac{z}{\phi(\lambda; t-1)} dz \\ &= -\lambda \phi(\lambda; t-1) + 2 \int_0^\lambda \phi(z; t-1) dz. \end{aligned}$$

Let λ^{BMAP} be the unique positive root of $P(\lambda)$. Then, we conjecture that the BEC MAP threshold without miscorrection, $p_t^{\text{BMAP}} = \lambda^{\text{BMAP}} / \phi(\lambda^{\text{BMAP}}; t-1)$, is also the threshold of the spatially-coupled HDD decoder. One can use algebra to prove the following lemma.

Lemma 7. *There is a $T < \infty$ such that the BEC MAP threshold $p_t^{\text{BMAP}} \geq 2t - 2$ for all $t \geq T$.*

VI. NUMERICAL RESULTS AND COMPARISON

We first introduce the definition the ϵ -redundancy optimality of a code ensemble.

Definition 8. Let $C(p)$ be the capacity of a BSC(p). For any $\epsilon > 0$, a code ensemble with rate R and threshold p^* is called ϵ -redundancy achieving if

$$\frac{1 - C(p^*)}{1 - R} \geq 1 - \epsilon.$$

Let $n_\nu \triangleq 2^\nu - 1$. The following lemma shows that, a sequence of ensembles, parameterized by $\nu \in \mathbb{Z}_+$, with rate $R_\nu = 1 - \frac{2\nu t}{n_\nu}$ and threshold $p_\nu^* = \frac{2t}{n_\nu}$ is ϵ -redundancy achieving when ν is large.

Lemma 9. *Consider a sequence of BSCs with error probability $2tn_\nu^{-1}$ for fixed t and increasing ν . Then, the ratio of $1 - C(2tn_\nu^{-1})$ and $2tn_\nu^{-1}$ goes to 1 as $\nu \rightarrow \infty$. That is*

$$\lim_{\nu \rightarrow \infty} \frac{1 - C(2tn_\nu^{-1})}{2tn_\nu^{-1}} = 1. \quad (14)$$

Proof: Recall that the capacity of the BSC(p) is $1 - H(p)$, where $H(p) = -p \log_2(p) - (1-p) \log_2(1-p)$ is the binary entropy function. The numerator of (14) can be written as

$$1 - C\left(\frac{2t}{n_\nu}\right) = H\left(\frac{2t}{n_\nu}\right) = \frac{2t \log_2 n_\nu}{n_\nu} \left(1 - \frac{\log_2\left(\frac{2t}{e}\right)}{\log_2 n_\nu} - O(n_\nu^{-1})\right). \quad (15)$$

By substituting (15) into the RHS of (14), we have

$$\frac{1 - C\left(\frac{2t}{n_\nu}\right)}{2tn_\nu^{-1}} = \frac{2tn_\nu^{-1} \log_2(n_\nu)}{2tn_\nu^{-1}} (1 - O(\nu^{-1})).$$

Then, the equality (14) follows since $\log_2(n_\nu) = \nu + o(1)$. ■

In the following numerical results, the iterative HDD threshold of (\mathcal{C}, m, L, w) spatially-coupled GLDPC ensemble with $L = 512$, and $w = 16$ are considered. In Table I, the thresholds of the ensembles are shown by computing the average number of error bits attached to a code-constraint node. For finite n , let $p_{n,t}^*$ be the iterative HDD threshold of the spatially-coupled GLDPC ensemble with $(n, k, 2t+1)$ binary primitive BCH component code, and $\tilde{p}_{n,t}^*$ be the iterative HDD threshold of the spatially-coupled GLDPC ensemble with $(n, k-1, 2t+2)$ even-weight BCH subcodes. Then, we define $a_{n,t}^* \triangleq np_{n,t}^*$ and $\tilde{a}_{n,t}^* \triangleq n\tilde{p}_{n,t}^*$ as the thresholds of the average number of error bits per component code. When $n \rightarrow \infty$, we denote λ_t^* and $\tilde{\lambda}_t^*$ as the iterative HDD thresholds of the ensembles with primitive BCH component code and its even-weight subcode, respectively. Moreover, the threshold of HDD without miscorrection, $\hat{\lambda}_t^*$, is shown in Table I along with the EXIT upper bound on the BEC MAP threshold, $\hat{\lambda}_t^{\text{BMAP}}$.

As shown in Table I, the thresholds of the spatially-coupled ensemble with the primitive BCH codes or the even-weight BCH subcodes are quite close to $2t$ as t increases. The vanishing effect of miscorrection can be shown using Lemma 4. As t increases, both $\hat{Q}_n(i)$ and $\tilde{Q}_n(i)$ approach 0. Also in Table I, one sees that λ_t^* , $\tilde{\lambda}_t^*$ and $\hat{\lambda}_t^*$ for large t are close to each other because the miscorrection probability becomes negligible. One can also observe that λ_t^* , $\tilde{\lambda}_t^*$ and $\hat{\lambda}_t^*$ approach to $2t$ as t increases. , we propose the following conjecture.

Conjecture 10. For any $\epsilon > 0$, there is a t where $\hat{\lambda}_t^* > 2t(1-\epsilon)$.

Conjecture 10 implies that one can choose t large enough and then N large enough so that $p_{n,t}^* > \frac{2t}{n}(1-\epsilon)$ for all $n \geq N$.

Proposition 11. If Conjecture 10 holds, then, for any $\epsilon > 0$, there exist a $T(\epsilon)$ and an $N(\epsilon, T)$ such that, when the component codes are t -error correcting BCH codes with $t \geq T(\epsilon)$ and $n \geq N(\epsilon, T)$, the (\mathcal{C}, m, L, w) GLDPC spatially-coupled ensemble with iterative HDD is ϵ -redundancy achieving.

Sketch of proof : The proposition follows from Lemma 9 and the definition of scaled DE. ■

Table I
THE ITERATIVE HDD THRESHOLDS OF $(\mathcal{C}, \infty, 512, 16)$
SPATIALLY-COUPLED GLDPC ENSEMBLE WITH BINARY PRIMITIVE BCH
CODES AND ITS EVEN-WEIGHT SUBCODES

t	3	4	5	6	7
$a_{255,t}^*$	5.432	7.701	9.818	11.86	13.87
$a_{511,t}^*$	5.417	7.665	9.811	11.86	13.85
$a_{1023,t}^*$	5.401	7.693	9.821	11.87	13.88
λ_t^*	5.390	7.688	9.822	11.91	13.93
$\tilde{a}_{255,t}^*$	5.610	7.752	9.843	11.88	13.87
$\tilde{a}_{511,t}^*$	5.570	7.767	9.811	11.86	13.85
$\tilde{a}_{1023,t}^*$	5.606	7.765	9.841	11.88	13.88
$\tilde{\lambda}_t^*$	5.605	7.761	9.840	11.90	13.92
$\hat{\lambda}_t^*$	5.735	7.813	9.855	11.91	13.93
$\hat{\lambda}_t^{\text{BMAP}}$	5.754	7.843	9.896	11.93	13.95

VII. CONCLUSION

The iterative HDD of GLDPC ensembles, based on on t -error correcting block codes, is analyzed with and without spatial coupling. Using DE analysis, noise thresholds are computed for a variety of component codes and decoding assumptions. In particular, the case of binary primitive BCH component-codes is considered along with their even-weight subcodes. For these codes, the miscorrection probability is characterized and included in the DE analysis. Scaled DE recursions are also computed for the high-rate limit. When miscorrection is neglected, the resulting recursion for the basic ensemble matches the results of [12], [13]. Upper bounds on the BEC MAP threshold of these codes are also computed and the phenomenon of threshold saturation is observed empirically. It is conjectured that threshold saturation does indeed occur for the scaled DE and, assuming this is true, it is shown that that iterative hard-decision decoding (HDD) of the spatially-coupled GLDPC ensemble can approach capacity in high-rate regime. Finally, numerical results are presented that both support this conjecture and demonstrate the effectiveness of these codes for high-speed communication systems.

REFERENCES

- [1] C. E. Shannon, "A mathematical theory of communication," *The Bell Syst. Techn. J.*, vol. 27, pp. 379–423, 623–656, July / Oct. 1948.
- [2] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo-codes," in *Proc. IEEE Int. Conf. Commun.*, vol. 2. Geneva, Switzerland: IEEE, May 1993, pp. 1064–1070.
- [3] M. G. Luby, M. Mitzenmacher, M. A. Shokrollahi, and D. A. Spielman, "Efficient erasure correcting codes," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 569–584, Feb. 2001.
- [4] T. J. Richardson, M. A. Shokrollahi, and R. L. Urbanke, "Design of capacity-approaching irregular low-density parity-check codes," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 619–637, Feb. 2001.
- [5] P. Bender, P. Black, M. Grob, R. Padovani, N. Sindhushayana, and A. J. Viterbi, "CDMA/HDR: A bandwidth-efficient high-speed wireless data service for nomadic users," *IEEE Commun. Magazine*, vol. 38, no. 7, pp. 70–77, July 2000.
- [6] C. Douillard, M. Jezequel, C. Berrou, N. Brengarth, J. Tousch, and N. Pham, "The turbo code standard for DVB-RCS," in *Proc. Int. Symp. on Turbo Codes & Related Topics*, 2000, pp. 535–538.
- [7] N. Abramson, "Cascade decoding of cyclic product codes," *IEEE Trans. Commun. Tech.*, vol. 16, no. 3, pp. 398–402, 1968.
- [8] R. M. Pyndiah, "Near-optimum decoding of product codes: Block turbo codes," *IEEE Trans. Commun.*, vol. 46, no. 8, pp. 1003–1010, Aug. 1998.
- [9] I.-T. R. G.975.1, "Forward error correction for high bit-rate DWDM submarine systems," 2004.
- [10] D. Truhachev, M. Lentmaier, and K. Zigangirov, "On braided block codes," in *Proc. IEEE Int. Symp. Inform. Theory*. IEEE, 2003, p. 32.
- [11] A. Feltstrom, D. Truhachev, M. Lentmaier, and K. Zigangirov, "Braided block codes," *IEEE Trans. Inform. Theory*, vol. 55, no. 6, pp. 2640–2658, 2009.
- [12] J. Justesen, K. Larsen, and L. Pedersen, "Error correcting coding for OTN," *IEEE Commun. Magazine*, vol. 48, no. 9, pp. 70–75, 2010.
- [13] J. Justesen, "Performance of product codes and related structures with iterated decoding," vol. 59, no. 2, pp. 407–415, 2011.
- [14] T. J. Richardson and R. L. Urbanke, *Modern Coding Theory*. Cambridge University Press, 2008.
- [15] S. Kudekar, T. Richardson, and R. Urbanke, "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC," *IEEE Trans. Inform. Theory*, vol. 57, no. 2, pp. 803–834, 2011.
- [16] M. Kim and J. Lee, "Undetected error probabilities of binary primitive BCH codes for both error correction and detection," *IEEE Trans. Commun.*, vol. 44, no. 5, pp. 575–580, 1996.